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Translated by A. Y.

PERTURBATION OF NATURAL SMALL VIBRATION FREQUENCIES UPON INTRODUCTION OF DAMPING

PMM Vol. 33, №2, 1969, pp. 328-330

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(Received April 1, 1968)

It is known that the complex natural frequencies $p_n = i\omega_n$ of a vibrating system take the form $p_n' = -\alpha_n + i\omega_n', \alpha_n \geqslant 0$ upon the introduction of damping. It can be shown that under some condition the imaginary part of the complex frequency hance varies thus,

$$\omega_N' \leqslant \omega_N \text{ for } \omega_N > 0, \quad \omega_N' \geqslant \omega_N \text{ for } \omega_N < 0$$
$$|\omega_N| = \max_n |\omega_n| \tag{1}$$

The proof of the inequalities (1) follows from this lemma.

Lemma. Let $A > 0, B \geqslant 0, R \geqslant 0$ be self-adjoint $n \times n$ matrices, where the condition

$$(Rx, x)^2 < 4(Ax, x)(Bx, x)$$

(weak damping) is satisfied. Let $p_n = i\omega_n$ be the roots of the equation

$$\det(p^2 A + B) = 0 \tag{2}$$

and p_n' the roots of the equation

$$p^2 A + pR + B) = 0 \tag{3}$$

Let $|p_N| = \max_n |p_n|$. Then $p_N' = -\alpha_N' + i\omega_N'$, where $\alpha_N' \geqslant 0$, and the inequalities (1) are satisfied. Here p_N' denotes the root of (3) for which $|\omega_N'| = \max_n |\omega_N'|$.

Proof. Let x_N' be the eigenvector corresponding to the eigennumber p_n', that is

$$(p_N'^2 A + p_N' R + B) x_{N'} = 0 \tag{4}$$

Then

$$p_N'^2 (Ax_N', x_N') + p_N' (Rx_N', x_N') + (Bx_N', x_N') = 0$$

Hence

$$p_N' = \frac{-(Rx_N', x_N') \pm i\sqrt{4(Bx_N', x_N')(Ax_N', x_N') - (Rx_N', x_N')^2}}{2(Ax_N', x_N')} \tag{5}$$

Since $R \geqslant 0, A > 0$, then $\alpha_N' \geqslant 0$. Furthermore

$$\omega_N' = \left(\frac{(Bx_N', x_N')}{(Ax_N', x'_N)} - \frac{(Rx_N', x_N')^2}{4(Ax_N', x_N')^2}\right)^{1/2} \tag{6}$$

From the minimax principle it follows that

$$\omega_N^2 = \sup_x \frac{(Bx, x)}{(Ax, x)} \geqslant \frac{(Bx_N', x_N')}{(Ax_N', x_N')} \quad (\omega_N^2 = \max_n \omega_n^2). \tag{7}$$

Inequalities (1) follow from (6) and (7). From the equation of motion

$$Au'' + Ru' + Bu = 0 \tag{8}$$

we obtain the equation

$$p^2 Ax + pRx + Bx = 0 \tag{9}$$

by means of substitution $u = e^{pt}x$, where x is independent of t.

The natural frequencies of this equation are found from the relationship (3), hence the inequalities (1) result from the lemma.

Note 1. As is seen from the proof of the lemma, the condition $R \geqslant 0$ is utilized just to prove the inequality $\alpha_n' \geqslant 0$. The inequalities (1) will hold for any R satisfying the weak damping condition, which conserves the vibrational nature of the process since only the square of the form (Rx_N', x_N') enters into (6).

Inequalities analogous to (1) are valid for continuous systems as well. For example, let us consider the equation of the vibrations of a finite elastic string. In this case the equation of motion is

$$u_{tt} + Ru_t + Bu = 0 \tag{10}$$

where B is a positive definite operator in the Hilbert space $H = L_2(0, l)$, and R is a linear operator describing friction.

For example, if a string fixed at the endpoints vibrates in a viscous medium, then R is the operator of multiplication by a positive constant, $B = -c^2(d^2(\ldots)/dx^2)$ and is defined as an operator in H by the boundary conditions $u(0) = u(l) = 0$.

Time is considered as a parameter. The operator B has the natural frequency spectrum

$$0 < \omega_1^2 \leqslant \omega_2^2 \leqslant \ldots \leqslant \omega_n^2 \leqslant \ldots, \quad \lim_{n\to\infty} \omega_n^2 = \infty$$

The operator B^{-1} has the values $\nu_n^2 = \omega_n^{-2}$ at points of the spectrum, so that $\max_n \nu_n^2 = \omega_1^{-2}$. The operator B^{-1} will be completely continuous in H, hence if R is a bounded operator, the operator $B^{-1}R$ is also completely continuous. If we put $u = e^{pt}v$, where $v \subset H$ is independent of t, then we obtain from (10)

$$(p^2 I + pR + B)v = 0. \tag{11}$$

Since the operator B^{-1} exists, and is bounded, then (11) is equivalent to the following:

$$p^2 B^{-1}v + pB^{-1}Rv + v = 0 \tag{12}$$

Since the operators B^{-1} and $B^{-1}R$ are completely continuous, the spectrum of (12) is discrete ([1], p. 30). For $R = 0$ the points of the spectrum of (12) are $p_n = i\omega_n$, where ω_n^2 are the eigennumbers of the operator B.

In order for the reasoning below to be analogous to that presented in the examination of the finite case, let us call the quantity $q = 1/p$ the complex natural frequency of Eq. (12). Then the frequency $q_1 = (i\omega_1)^{-1} = -i\nu_1$ is the largest complex frequency, in absolute value, of (12) for $R = 0$. Let us show that the inequality

$$\nu_1' \leqslant \nu_1 \tag{13}$$

holds for $R \neq 0$ for the perturbed frequency $q'_1 = -\alpha_1 - i\nu_1'$, and moreover, $\alpha_1 \geqslant 0$ for $R \geqslant 0$.

Reasoning as in the proof of the lemma, we deduce from (11)

$$q^2 (Bv, v) + q(Rv, v) + (v, v) = 0$$

Here (v, u) is the scalar product in H. Hence

$$q_1' = -\frac{(Rv, v)}{2(Bv, v)} - i\left(\frac{(v, v)}{(Bv, v)} - \frac{1}{4}\left|\frac{(Rv, v)}{(Bv, v)}\right|^2\right)^{1/2} \equiv -\alpha_1 - i\nu_1' \tag{14}$$

Since

$$\nu_1^2 = \max_v \frac{(v,\ v)}{(Bv,\ v)}$$

then $\nu_1 \leqslant \nu_1$.

If $R \geqslant 0$, then taking into account that $(Bv,\ v) > 0$, we obtain the inequality $\alpha_1 \geqslant 0$ from the definition of α_1. The proposed assertion is proved completely. Let us note that the sign in front of the root in (14) has been selected so that the equality $q_1' = q_1 = i\nu_1$ would hold for $R=0$.

Note 2. The following theorem is proved by the reasoning presented.

Theorem. Let B be a positive definite operator in the Hilbert space H, which has a completely continuous inverse operator B^{-1}. Let R be a linear operator, where $B^{-1}R$ is a completely continuous operator. Let $\nu_1^2 \geqslant \nu_2^2 \geqslant \ldots > 0$ be the eigennumbers of the operator B^{-1}. Then the imaginary part of any eigennumber of the operator $q^2B + qR + I$ is not greater than ν_1, but if the operator R is nonnegative, then the real parts of the eigennumbers of the operator $q^2B + qR + I$ are nonpositive.

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Translated by M. D. F.

OSCILLATIONS OF SYSTEMS WITH TIME - DEPENDENT PARAMETERS

PMM Vol. 33, №2, 1969, pp. 331-337

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(Received September 3, 1968)

There are several approximate methods [1−4] available for solving practical problems having to do with oscillatory systems whose parameters vary with time. The procedure for analyzing such systems proposed in the present paper is based on the analogy between parametric and forced oscillations in a certain nominal oscillator with parameters chosen in a certain special way. Our approach, which closely resembles the idea behind the WKB (Wentzel-Kramers-Brillouin) method [3−5], provides increased opportunities for constructing effective approximate solutions of problems of the indicated class.

1. We begin by considering the following linear second-order differential equation to which many problems of applied dynamics can be reduced [1 and 5]:

$$\ddot{q} + 2n(t)\dot{q} + k^2(t)q = F(t) \tag{1.1}$$

The Euler substitution reduces Eq. (1.1) to the form

$$y^{\cdot\cdot} + p^2(t)\, y = Q(t), \qquad (p^2 = k^2 - n^2 - \dot{n}) \tag{1.2}$$

$$y = q \exp\left[\int_0^t n(t)\, dt\right], \qquad Q = F \exp\left[\int_0^t n(t)\, dt\right]$$

The solution of the homogeneous equation corresponding to Eq. (1.2) is obtainable in